

A Class of Elastic-Plastic Constitutive Laws for Brittle Rock

J. W. RUDNICKI, *Department of Civil Engineering,
Northwestern University, Evanston, Illinois 60201*

Synopsis

The inelastic behavior of fissured rock masses is due primarily to microcracking from the tips of preexisting fissures and frictional sliding on fissure surfaces. Consequently, the macroscopic inelastic response is inhibited by an increase of hydrostatic compression and exhibits volume change and strain softening. By generalizing the type of laws often used in metal plasticity, Rudnicki and Rice introduced a class of simple constitutive laws that incorporate these features and is useful for studying the inception of rupture. An important aspect of this generalization is that normality of the inelastic strain increment vector to a yield surface in stress space, as assumed in classical metal plasticity, is not appropriate. More detailed consideration of the preferential activation of sliding on differently oriented fissure surfaces during a program of loading suggests that, although this class of laws will be suitable for describing loading in which stress components increase nearly in proportion to a single parameter, they will be inadequate for describing abrupt changes in the pattern of deformation. An approximate remedy for this inadequacy can be interpreted as deformation (or total strain) theory of plasticity. Mechanical coupling between diffusion of an infiltrating pore fluid, for example, ground water, and deformation can also be included by replacing the hydrostatic stress σ by the effective stress $\sigma - \zeta p$, where p is the pore fluid pressure and ζ satisfies $0 < \zeta \leq 1$ for elastic deformation and $\zeta = 1$ for inelastic deformation typical of brittle rock. This coupling causes the response to be time dependent even when the response of the matrix material is time independent. Specifically, the response is stiffer for load alterations that are rapid by comparison to the diffusion time of pore fluid than for those that allow time for equilibration of pore fluid pressure among neighboring material elements.

INTRODUCTION

For temperatures and pressures that are typical of the earth's crust, the primary mechanism of inelastic deformation in rock is microcrack growth. Even though all the principal stresses in the earth's crust are likely to be compressive, microcracking can re-

sult from local tensile stresses caused by mismatches in elastic properties of constituent minerals or induced near the tips of preexisting fissures by frictional slip on the fissure surfaces. Such microcrack growth is very different from the microscale mechanisms typical of metal plasticity, e.g., dislocation motion and single crystal slip. Nevertheless, an approach that has proven useful in metal plasticity can be adapted to model the inelastic response of brittle rocks. It is, of course, necessary to take proper account of differences in macroscopic behavior that reflect the different underlying microstructural mechanisms. These include inelastic volume change accompanying inelastic shear, dependence of inelastic deformation on mean normal stress, strain softening, and degradation of elastic moduli caused by inelastic deformation.

This article reviews and elaborates upon a constitutive framework proposed by Rudnicki and Rice¹ to model the time-independent behavior of brittle rock. This framework was introduced in the context of a study of the inception of localization of deformation as a bifurcation from homogeneous deformation and, consequently, was not meant to be a comprehensive description of brittle rock behavior. Nevertheless, the formulation does include the features of brittle rock behavior mentioned above. Moreover, the parameters of the constitutive law can be determined by experiment, the law is simple enough to be used in the solution of boundary value problems, and it is suitable for arbitrary deformation states. This constitutive law is, however, one of a class of laws that is known to be inadequate in describing the response to deformation increments causing abrupt changes in the ratios of stress components. A remedy for this inadequacy suggested by Rudnicki and Rice¹ is described and interpreted in terms of a deformation theory of plasticity. Finally, a brief description is given of the modification necessary for including the mechanical effects of coupling between deformation and diffusion of an infiltrating pore fluid.

CONSTITUTIVE LAW FOR A SIMPLE DEFORMATION STATE

This section follows the development of Rudnicki and Rice,¹ based on the earlier treatment of Rice,² and introduces the constitutive law by first considering the case of a material element

subjected to a combination of pure shear stress τ and hydrostatic stress σ (taken to be positive in compression). This element may have undergone some past history of inelastic deformation which is reflected in the current microstructure, e.g., the extent of microcracking, frictional slip, void growth, etc., and the goal is to describe the next increment of deformation. In general, the next deformation increment can involve both elastic and further inelastic response. The increments of shear strain, $d\gamma$, and volume strain, $d\epsilon$ (positive for volume increase) are given by

$$\begin{aligned}d\gamma &= d\tau/G + d^p\gamma, \\d\epsilon &= -d\sigma/K + d^p\epsilon,\end{aligned}\tag{1}$$

where G and K are the incremental elastic shear and bulk moduli, respectively. The second term in each equation denotes the inelastic contribution and these terms are dropped for elastic unloading. If $d\sigma = 0$, the inelastic increment of shear strain can be expressed as

$$d^p\gamma = d\tau/h,\tag{2}$$

where the inelastic modulus h is termed hardening if positive or softening if negative and is related to the slope of the τ vs. γ curve at constant σ by

$$\frac{d\tau}{d\gamma} = \frac{h}{1 + h/G}\tag{3}$$

as shown in Figure 1. An increase in hydrostatic stress σ inhibits further inelastic deformation due to microcracking and frictional slip. Consequently, τ vs. γ curves for different constant values of σ are as shown schematically in Figure 2(a). Unloading elastically from one curve to another locates points on a yield surface shown in Figure 2(b). For values of τ and σ on the yield surface, this surface forms the boundary between increments ($d\sigma$ and $d\tau$) that tend to cause further inelastic deformation and those that tend to cause elastic unloading. Hence, for $h > 0$, increments for which $d\tau \leq \mu d\sigma$ correspond to elastic unloading and those for which $d\tau > \mu d\sigma$ correspond to further inelastic deformation, where μ is the local slope of the yield surface and the inequalities are reversed for $h < 0$. The parameter μ is called a friction coefficient although the slope of the yield surface in the τ vs. σ plane may simply

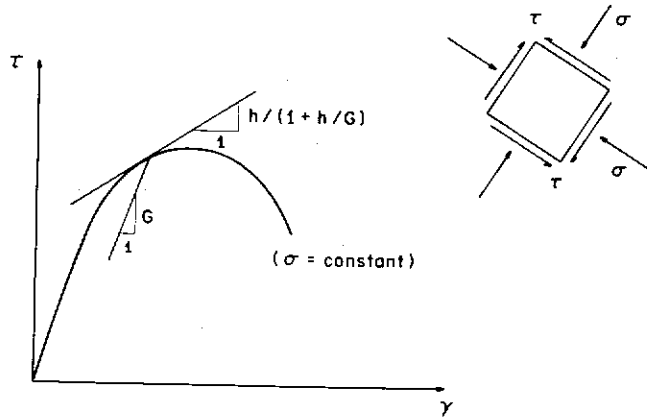


Fig. 1. Shear stress τ versus shear strain γ for constant hydrostatic stress σ . The slope of the curve is related to the hardening (softening) modulus h as shown. The elastic shear modulus for unloading is G .

reflect the decreasing range of orientations for which tensile cracking is possible as σ is increased rather than being specifically related to frictional sliding on fissure surfaces.³ When both $d\tau$ and $d\sigma$ are nonzero the expression for $d^p\gamma$ becomes

$$d^p\gamma = (d\tau - \mu d\sigma)/h. \quad (4)$$

In brittle rocks, inelastic volume change accompanies inelastic shear and, consequently, it is plausible to assume that

$$d^p\epsilon = \beta d^p\gamma, \quad (5)$$

where β is a dilatancy factor. Rocks having low initial porosities typically exhibit volume increase⁴ even when all principal stresses are compressive. Such volume increase, or dilatancy, is apparently due to microcracking and, possibly, to uplift in frictional sliding over asperities on fissure surfaces. In rocks of high initial porosity dilatancy can be preceded by inelastic compaction due to shear-induced collapse of pores.⁵ During compaction, the dilatancy factor is negative.

The full strain increments for inelastic response are as follows:

$$d\gamma = d\tau/G + (d\tau - \mu d\sigma)/h, \quad (6)$$

$$d\epsilon = -d\sigma/K + \beta(d\tau - \mu d\sigma)/h. \quad (7)$$

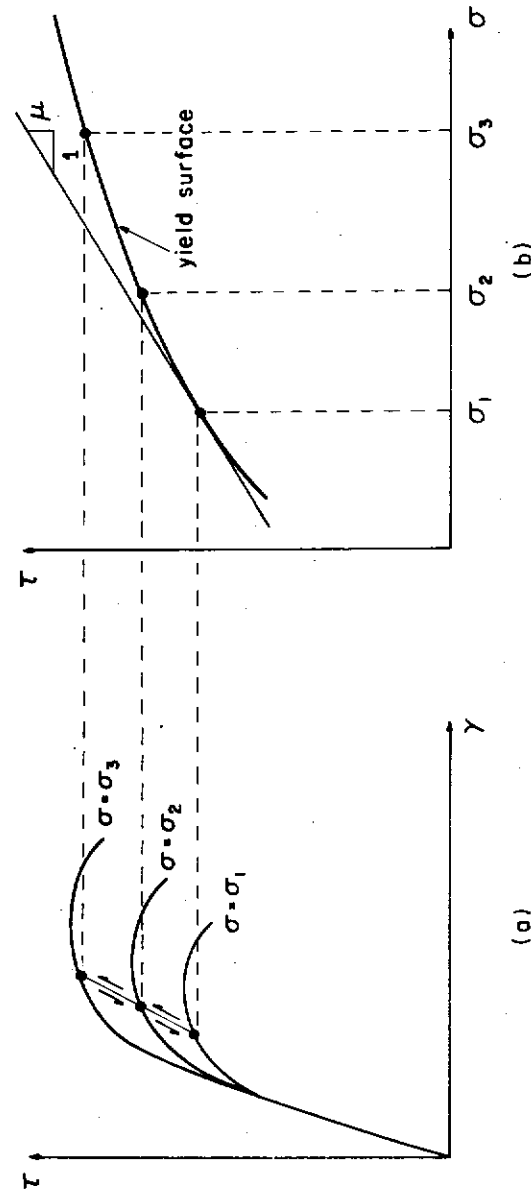


Fig. 2. (a) Shear stress versus shear strain curves at different values of hydrostatic stress $\sigma_1 < \sigma_2 < \sigma_3$. Points on different curves that can be reached by unloading locate points on the yield surface (b) in the τ vs. σ plane. The slope of the yield surface is a friction coefficient μ .

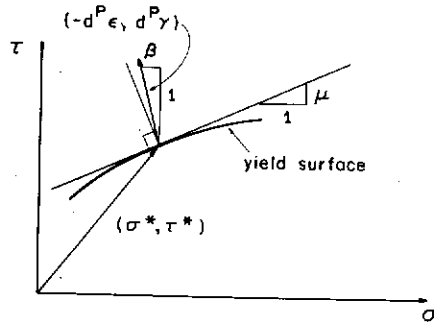


Fig. 3. The inelastic increment of strain is shown as a vector $(-d^P \epsilon, d^P \gamma)$ emanating from the current stress point (σ^*, τ^*) in the τ vs. σ plane. The inelastic strain increment vector is normal to the yield surface only if $\mu = \beta$.

In Figure 3, the inelastic strain increment is plotted as a vector in the τ vs. σ plane and, as indicated, this vector is normal to the yield surface only when $\beta = \mu$. In metal plasticity, the inelastic strain increment vector is generally assumed to be normal to the yield surface. Rice⁶ and Hill and Rice⁷ have shown that normality results when the microstructural mechanisms of inelastic deformation depend on the applied stress only through the thermodynamically work-conjugate force. For example, this is the case for dislocation slip which depends on the applied stress only through the resolved shear stress, at least when elastic strains are small. However, inelastic mechanisms that are frictional involve dependence on the normal stress as well as the resolved shear stress and, consequently, are not expected to obey the normality rule.

The notation in Eqs. (1), (2), (4), and (5) is that adopted by Hill and Rice⁷ and serves to emphasize that an *inelastic increment of strain*, for example, $d^P \gamma$, can result from degradation of elastic moduli as well as from *increments in inelastic strain*, for example, $d\gamma^P$. The distinction is illustrated by the ideal stress-strain curves shown in Figure 4. Figure 4(a) shows the idealized response of metals: The elastic moduli are unaffected by inelastic deformation and the inelastic strain is identified as the residual strain upon unloading. Figure 4(b) shows a stress-strain curve for an ideally brittle material of the type discussed by Rice⁸ for which inelastic deformation is due entirely to the extension of sharp-tipped Griffith cracks. In this case, the strain is fully recovered upon unloading to zero stress and there is no "inelastic strain."

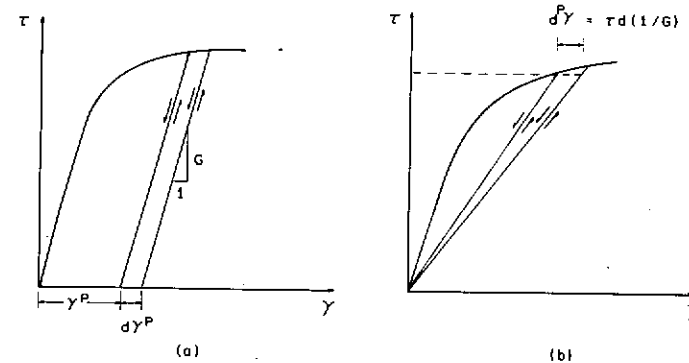


Fig. 4. Idealized stress-strain curves for a metal (a) and a brittle material in which the only inelastic deformation is due to growth of sharp-tipped cracks (b). In (a) the elastic modulus is unaffected by inelastic deformation and the inelastic strain remains after unloading. In (b), unloading returns the strain to zero, and the inelastic increment of strain is due entirely to the incremental increase in the compliance $(1/G)$.

There is, however, an increment of inelastic strain due to the degradation of elastic moduli by microcrack extension. The distinction between these two types of inelastic deformation is significant in the present context since much of the inelastic deformation of brittle rock results from microcrack extension.

GENERALIZATION TO ARBITRARY DEFORMATION STATES

The generalization of Eqs. (6) and (7) to other deformation states is, to a large extent, arbitrary. Most experiments in rock deformation are conducted in axisymmetric compression or extension and, consequently, provide little constraint on constitutive formulations for fully three-dimensional deformation. Moreover, there is still disagreement about the microstructural mechanisms primarily responsible for inelastic deformation in brittle rock⁹⁻¹² and modeling studies based on these mechanisms have not progressed to the point where they can guide the development of multi-axial constitutive laws. Under these circumstances, the simplest generalization is most appealing.

Rudnicki and Rice¹ adopt a generalization similar to that used to obtain the Prandtl-Reuss equations of metal plasticity.¹³ This

approach replaces the simple shear stress τ by so-called Mises equivalent stress:

$$\bar{\tau} = (1/2 \sigma'_{ij} \sigma'_{ij})^{1/2},$$

where $\sigma'_{ij} = \sigma_{ij} - (1/3) \sigma_{kk} \delta_{ij}$ is the deviatoric stress. A geometric interpretation of this replacement is that when the yield surface is plotted in a space with the principal stresses as coordinates, the projection on the plane $\sigma_{kk} = 0$ is a circle. The radius of the circle increases (decreases) with inelastic deformation if $h > (<) 0$. Hence, the inelastic response is isotropic at each stage of the deformation and formulations of this type are called "isotropic hardening." Also, the deviatoric components of the inelastic increments of strain are assumed to be proportional to the corresponding components of σ'_{ij} . This assumption requires that when the inelastic increment of strain is plotted in stress space as a vector emanating from the point on the yield surface corresponding to the current stress state, the projection on the plane $\sigma_{kk} = 0$ is perpendicular to the circular projection of the yield surface on this plane.

With these replacements and assumptions, the Eqs. (6) and (7) become

$$d\epsilon'_{ij} = \frac{d\sigma'_{ij}}{2G} + \frac{1}{2h} \frac{\sigma'_{ij}}{\bar{\tau}} \left(\frac{\sigma'_{kl} d\sigma_{kl}}{2\bar{\tau}} + \mu \frac{d\sigma_{kk}}{3} \right), \quad (8)$$

$$d\epsilon_{kk} = \frac{d\sigma_{kk}}{3K} + \frac{\beta}{h} \left(\frac{\sigma'_{kl} d\sigma_{kl}}{2\bar{\tau}} + \mu \frac{d\sigma_{kk}}{3} \right), \quad (9)$$

where the elastic response has been assumed to be isotropic and σ has been replaced by $-\sigma_{kk}/3$ (so that all stresses are now regarded as positive in compression). Alternatively, Eqs. (8) and (9) can be combined to yield

$$d\epsilon_{ij} = \frac{1}{2G} \left(d\sigma_{ij} - \frac{\nu}{1+\nu} d\sigma_{kk} \delta_{ij} \right) + \frac{1}{h} \left(\frac{\sigma'_{ij}}{2\bar{\tau}} + \beta \frac{\delta_{ij}}{3} \right) \left(\frac{\sigma'_{kl} d\sigma_{kl}}{2\bar{\tau}} + \mu \frac{d\sigma_{kk}}{3} \right), \quad (10)$$

where ν is Poisson's ratio, and the relation $K = 2G(1 + \nu)/3(1 - 2\nu)$ has been used. If $\mu = \beta = 0$, Eqs. (8) and (9) or (10) reduce to the form of the Prandtl-Reuss equations described by Hill.¹³ Al-

though these equations are not restricted to small strain, it is necessary to specify suitable materially objective measures of deformation and stress rate when the deformation involves principal axis rotation.¹⁴ A variety of choices are possible, but Rudnicki and Rice choose the rate of deformation D_{ij} where

$$D_{ij} = 1/2 (\partial v_i / \partial x_j + \partial v_j / \partial x_i) \quad (11)$$

is the symmetric part of the velocity gradient tensor and the Jaumann rate of Cauchy stress $\overset{\nabla}{\sigma}_{ij}$ defined by¹⁵

$$\overset{\nabla}{\sigma}_{ij} = \dot{\sigma}_{ij} - \Omega_{ik} \sigma_{kj} + \sigma_{ik} \Omega_{kj}, \quad (12)$$

where $\Omega_{ij} = (1/2) (\partial v_i / \partial x_j - \partial v_j / \partial x_i)$ is the spin tensor and $\dot{\sigma}_{ij}$ is the material derivative of stress. Thus, Eq. (10) becomes

$$D_{ij} = \frac{1}{2G} \left(\overset{\nabla}{\sigma}_{ij} - \frac{\nu}{1+\nu} \dot{\sigma}_{kk} \delta_{ij} \right) + \frac{1}{h} P_{ij} Q_{kl} \overset{\nabla}{\sigma}_{kl}, \quad (13)$$

where

$$P_{ij} = \sigma'_{ij} / 2\bar{\tau} + (1/3)\beta \delta_{ij}$$

is the "direction" of the inelastic portion of D_{ij} and

$$Q_{ij} = \sigma'_{ij} / 2\bar{\tau} + (1/3)\mu \delta_{ij}$$

is the "normal" to the yield surface. Of course, equivalent expressions could be derived in terms of other measures of stress rate and deformation. The inverted form of Eq. (13) is

$$\overset{\nabla}{\sigma}_{ij} = E_{ijkl} \left\{ D_{kl} - P_{kl} \frac{Q_{pq} E_{pqrs} D_{rs}}{h + Q_{pq} E_{pqrs} P_{rs}} \right\}, \quad (14)$$

where

$$E_{ijkl} = 2G \frac{\nu}{1-2\nu} \delta_{ij} \delta_{kl} + G(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})$$

is the tensor of elastic moduli. This form is sometimes convenient because the term $h + Q_{pq} E_{pqrs} P_{rs}$ is generally positive whereas h , the corresponding quantity in Eq. (13), may be positive or negative.

Equation (13) is the most general form of isotropic, incrementally linear (meaning that the D_{ij} depend linearly on the $\overset{\nabla}{\sigma}_{ij}$) law depending on the first and second stress invariants. Nevertheless,

there is no reason, except simplicity, for choosing this generalization of Eqs. (6) and (7) and others are possible. For example, Senseny, Fossum, and Pfeifle¹⁶ have introduced a constitutive law similar to Eq. (13) but having a yield surface with a hexagonal projection on the plane $\sigma_{kk} = 0$. This law does, however, depend on the third stress invariant.

It is worth emphasizing that the parameters μ , β , h , G , and K may depend on the deformation and even the past history of deformation although they are independent of the deformation or stress increments, except for the distinction between loading and unloading. Nemat-Nasser and Shokooh¹⁷ mistakenly interpret μ and β as constants in the formulation of Rudnicki and Rice and, as a consequence, claim to derive a more general formulation. In fact, the constitutive law discussed by them is one of the class introduced by Rudnicki and Rice and corresponds to choosing a specific dependence on deformation for parameters equivalent to μ , β , and h . With this choice Nemat-Nasser and Shokooh¹⁷ do, however, achieve good agreement between the predictions of the constitutive law and results of the axisymmetric compression tests on Ottawa sand and crushed Westerly granite at different confining pressures. In practice, some of the parameters in Eq. (13) change much less with deformation than others and, depending on the application, these may be taken as constant. For example, μ and β for brittle rocks are typically in the ranges 0.4–0.6 and 0.2–0.4 and change by only a factor of 2 or 3 during loading to peak stress in axisymmetric compression.^{1,2}

Despite its simplicity, Eq. (13) has proven to be useful as a framework for reporting the results of some experiments. Senseny, Fossum, and Pfeifle¹⁶ have determined the parameters of Eq. (13) for two low-porosity evaporite rocks, anhydrite and polyhalite, and Butcher¹⁸ has used the law to describe the time-independent behavior of rock salt in triaxial compression. It should, however, be noted that virtually all the observations are for axisymmetric compression and there has been no systematic study to assess the validity of Eq. (13) for a variety of deformation states. Specifically, the use of Mises equivalent stress implies a strong dependence of inelastic deformation on the intermediate principal deviatoric stress. Except for some studies of Mogi (quoted in Shimamoto¹⁹) with a cubical triaxial machine, there appears to have been no work done in this area. In addition to applications to rock, Eq. (13) has been used to describe the re-

sponse of metals when void growth and nucleation or strength-differential effects give rise to pressure dependence of inelastic behavior or inelastic volume change.²⁰

YIELD SURFACE VERTEX FORMULATION

Constitutive equations of the type (13) are known to be most suitable for loadings in which the components $\bar{\sigma}_{ij}$ increase roughly in proportion to one another, i.e., proportional loading. There is, however, substantial evidence from the testing of metals²¹ that such a formulation overestimates the stiffness of response when the loading involves an abrupt change in the ratio of components $\bar{\sigma}_{ij}$. The reason is that in Eq. (13) the ratios of the D_{ij}^p , the inelastic portions of the D_{ij} , are fixed according to the corresponding ratios of the P_{ij} . Consequently, the direction of D_{ij}^p is fixed by the current stress state and is independent of the direction of $\bar{\sigma}_{ij}$, a feature that is common to all elastic-plastic formulations that employ a smooth yield surface. Hill²² has, however, argued that when macroscopic inelasticity is the result of mechanisms that are heterogeneous on the microscale, a vertex is to be expected on the current yield surface. Moreover, microstructural models for brittle rock based on frictional sliding on fissure surfaces^{1,12} also indicate vertex formation. Although experiments cannot distinguish between an actual vertex and a sharply rounded nose, the results of experiments on metals do indicate dependence of the direction of D_{ij}^p on $\bar{\sigma}_{ij}$. Despite absence of comparable evidence for rocks, this defect of the smooth yield surface formulation is likely to persist.

To illustrate the significance of vertex formation it is convenient to specialize Eq. (13) to a form appropriate to planar, biaxial deformation. Thus, assuming $\sigma_{13} = \sigma_{23} = \sigma_{12} = 0$ and $\sigma_{33} = (1/2)(\sigma_{11} + \sigma_{22})$, the response for $\bar{\sigma}_{12}$, $\bar{\sigma}_{22}$, and $\bar{\sigma}_{11}$ nonzero is as follows:

$$D_{11} - D_{22} = (\bar{\sigma}_{11} - \bar{\sigma}_{22})/2G + \delta Q_{kl} \bar{\sigma}_{kl}/h, \quad (15)$$

$$D_{11} + D_{22} = (\bar{\sigma}_{11} + \bar{\sigma}_{22})/2M + 2\beta Q_{kl} \bar{\sigma}_{kl}/3h, \quad (16)$$

$$D_{12} = \bar{\sigma}_{12}/G, \quad (17)$$

where $\delta = \text{sign}(\sigma'_{11})$, $M = G(1 + \nu)/(1 - \nu)$ and

$$Q_{kl} \bar{\sigma}_{kl} = \delta(\bar{\sigma}_{11} - \bar{\sigma}_{22})/2 + \mu(\bar{\sigma}_{11} + \bar{\sigma}_{22})/3. \quad (18)$$

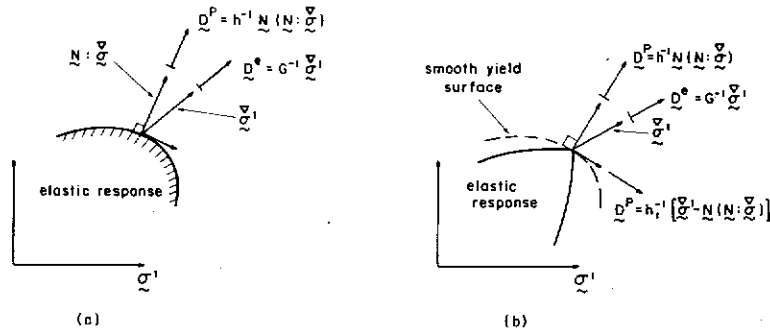


Fig. 5. The response to a deviatoric stress increment $\bar{\sigma}'$ is compared for a smooth yield surface idealization (a) and a yield surface with a sharp vertex. The normal to the smooth yield surface is $N_{ij} = \sigma'_{ij}/2\tau$ and $N:\bar{\sigma} = N_{kl}\bar{\sigma}_{kl}$. Rudnicki and Rice¹ approximate the response at a yield surface vertex by adding the term $h_1^{-1}[\bar{\sigma}' - N(N:\bar{\sigma})]$ to D^p .

Continued biaxial loading ($\bar{\sigma}'_{11} \neq 0$, $\bar{\sigma}'_{22} \neq 0$, $\bar{\sigma}'_{12} = 0$) is accompanied by inelastic deformation governed by the modulus h . For loading that involves an increment of shear ($\bar{\sigma}'_{12} \neq 0$) and, hence, a deviation from the previous pattern of deformation, this increment is governed by the elastic shear modulus. Because h becomes much less than G after some amount of inelastic deformation, the response to an increment of shear is much stiffer than the response to continued biaxial loading. This type of behavior, which is typical of the smooth yield surface idealization, is depicted geometrically in Figure 5(a). For convenience in illustration μ and β have been assumed to be zero, so that $Q_{ij} = P_{ij} = N_{ij}$, where $N_{ij} = \sigma'_{ij}/2\tau$. Note that any stress increment for which $N_{kl}\bar{\sigma}_{kl} = 0$, that is, an increment tangential to the yield surface, elicits purely elastic response. In contrast, the response at a yield surface vertex is shown in Figure 5(b). Stress increments in a direction tangential to what would be the smooth yield surface now cause inelastic deformation and only those increments directed back into the cone of the vertex cause purely elastic response. Moreover, the D^p_{ij} are constrained only to lie with the cone formed by normals to the sides of the vertex. Rudnicki and Rice used a simple approximation to the response at a vertex which, for the specialization to biaxial stressing, adds the term $\bar{\sigma}'_{12}/2h_1$ to Eq. (17). Because h_1 is an inelastic modulus that typically satisfies $h \ll h_1 < G$, this modification makes the response to an increment of

shear less stiff. The corresponding modification of the full Equation (13) is to append the term

$$\frac{1}{2h_1} \left(\bar{\sigma}'_{ij} - \frac{\sigma'_{ij}\sigma'_{kl}}{2\tau^2} \bar{\sigma}_{kl} \right). \quad (19)$$

Note that this term leaves D_{kk} unaltered (for reasons explained by Rudnicki and Rice) and vanishes when the stress increment continues the same pattern of stressing, that is, when $\bar{\sigma}'_{ij} \propto \sigma'_{ij}$.

In the next section, it will be shown, following an analysis by Stören and Rice,²³ that under some circumstances h_1 can be interpreted as the secant modulus for the shear stress versus shear strain curve. The simple approximation (19) is expected to be most suitable when the deviation in the pattern of loading is not too great. In general, the relation between $\bar{\sigma}'_{ij}$ and D_{ij} will not be linear at a yield surface vertex and a more elaborate formulation like the phenomenological corner theory proposed by Christoffersen and Hutchinson²⁴ for metal plasticity is necessary.

Mehrabadi and Cowin,²⁵ by generalizing Spencer's²⁶ theory for planar deformation of ideal granular materials, have arrived at a constitutive law that is identical to (19) in the rigid plastic limit, i.e., $G \rightarrow \infty$. However, in the Mehrabadi and Cowin approach, the modulus h_1 is given explicitly by

$$h_1 = \bar{\tau} (1 - \mu\beta)/(\beta - \mu).$$

Because μ is often greater than β , h_1 can be inherently negative. Although there is no reason to exclude this possibility, the equations governing deformation in such a material are hyperbolic at the onset of inelastic deformation and, hence, immediately admit the possibility of localized shear deformation.

DEFORMATION THEORY OF PLASTICITY

Deformation theory of plasticity attempts to describe the constitutive response by means of a relation between the total strain and the stress rather than between increments of strain and stress. Although the results of this approach can be made to agree with an incremental theory, like that described in the last section, for proportional loading, this theory is essentially a nonlinear elastic theory and, hence, is obviously unsuitable if unloading occurs. Nevertheless, Budiansky²⁷ has demonstrated that

deformation theory is appropriate for loading that does not deviate too severely from proportional loading. Furthermore, because the direction of the inelastic strain increments predicted by the deformation theory does depend on the direction of the stress increment, this approach is a better approximation to the response at a yield surface vertex than the incremental theory. As a result, the use of deformation theory in analyses of buckling,²⁸ and other problems in which the pattern of deformation undergoes an abrupt change, yields predictions more in accord with experiments than the incremental theory.

A deformation theory that agrees with the incremental theory (10) for proportional loading is as follows:

$$\epsilon_{ij}^p = F(\bar{\tau} + \mu\sigma_{kk}/3) \left(\frac{\sigma'_{ij}}{2\bar{\tau}} + \beta \frac{\delta_{ij}}{3} \right), \quad (20)$$

where strains are considered to be small, μ and β are now interpreted as constants, and, because loading and unloading occur along the same curve, there is no distinction between increments of inelastic strain and inelastic strain increments. An interpretation of the significance of the function F is revealed by specializing to pure shear: $\sigma_{12} = \tau$, $2\epsilon_{12}^p = \gamma^p$, and all other components zero. In this case, Eq. (20) reduces to $\gamma^p = F(\tau)$, and F can be expressed in terms of the secant modulus for the τ vs. γ^p curve h_{sec} as follows:

$$h_{sec} = \tau/F(\tau). \quad (21)$$

The tangent modulus h is given by

$$h = (dF/d\tau)^{-1}. \quad (22)$$

When $\mu = \beta = 0$, Eq. (20) reduces to the familiar form of $J_2 (= \bar{\tau})$ deformation theory.²⁷ Note, however, that, unless $\beta = \mu$, Eq. (20) is not hyperelastic or Green elastic²⁹ meaning that it is not possible to write Eq. (20) in the form

$$\sigma_{ij} = \partial w / \partial \epsilon_{ij}^p,$$

where w is an energy function.

The incremental form of Eq. (20) can be obtained by differentiation and the result is

$$d\epsilon_{ij}^p = \frac{1}{h} P_{ij} Q_{kl} d\sigma_{kl} + \frac{1}{2h_{sec}} (d\sigma'_{ij} - \sigma'_{ij} \sigma'_{kl} d\sigma_{kl} / 2\bar{\tau}^2), \quad (23)$$

where Eqs. (21) and (22) have been used and Q_{ij} and P_{ij} are defined following Eq. (13). Comparing the final term of Eq. (23) with (19) reveals the interpretation of h_1 as a secant modulus. Stören and Rice,²³ Hutchinson and Neale,³⁰ and Christoffersen and Hutchinson²⁴ have discussed finite strain generalizations of the deformation theory with $\mu = \beta = 0$.

EFFECTS OF PORE FLUID PRESSURE

When rock masses are fluid saturated, it is necessary to include the effects of interaction between the pore fluid pressure and the deformation. Although both chemical and mechanical effects can be important, only a brief account of a formulation for including the mechanical effects of the pore fluid will be given here. Furthermore, attention will be restricted to cases for which the mechanical behavior of the pore fluid can be characterized by a scalar pressure. Specifically excluded from consideration are very rapid deformations, for example, on the time scale of elastic wave propagation, which do not allow time for equilibration of pore fluid pressure among neighboring fissures comprising a "point" in the continuum idealization. Although there exists no comprehensive formulation for this case, the problem has been addressed by O'Connell and Budiansky³¹ and Cleary.³²

The so-called effective stress principle states that the mechanical effect of the pore fluid can be incorporated by replacing the hydrostatic stress by the effective stress, a linear combination of the pore fluid pressure and the hydrostatic stress. For isotropic materials, the effective stress has the form

$$\bar{\sigma}_{ij} = \sigma_{ij} + \zeta p \delta_{ij}, \quad (24)$$

where p is the pore fluid pressure measured from some ambient level and $0 < \zeta \leq 1$. Nur and Byerlee³³ (also, see ref. 34) have shown that for elastic deformation

$$\zeta = 1 - K/K'_s, \quad (25)$$

where K is the bulk modulus of the matrix material and K'_s is another bulk modulus. Under modest restrictions, K'_s can be identified with the bulk modulus of the solid constituents. Carroll³⁵ has discussed the appropriate form of the effective stress for elastic deformation of anisotropic material. Rice³⁶ has shown that

when inelastic deformation results from the extension of sharp-tipped fissures and from sliding on fissure surfaces at sharp asperity contacts $\zeta = 1$. Of course, other forms of the effective stress may be appropriate for inelastic deformation arising from other types of mechanisms or for describing the effects of pore pressure on properties such as permeability. Substituting the effective stress into Eq. (13) yields

$$D_{ij} = \frac{1}{2G} \bar{\sigma}'_{ij} + \frac{1}{3K} \left[\frac{\dot{\sigma}_{kk}}{3} + \left(1 - \frac{K}{K'_s}\right) \dot{p} \right] \delta_{ij} + \frac{1}{h} P_{ij} \left[\frac{\sigma'_{kl}}{2\bar{\tau}} \bar{\sigma}_{kl} + \mu \left(\frac{\dot{\sigma}_{kk}}{3} + \dot{p} \right) \right], \quad (26)$$

where the relation $K = 2G(1 + \nu)/3(1 - 2\nu)$ has been used.

If the spatial distribution and time dependence of the pore fluid pressure can be determined, Eq. (26) suffices to yield the effect of the pressure on the deformation. In general, however, the pore fluid pressure must be determined simultaneously with the stress and deformation. Consequently, additional constitutive equations are needed for the fluid component. The rate of change of the fluid mass content per unit volume m can be expressed as

$$\dot{m} = \frac{\partial}{\partial t} (\rho v) = \dot{\rho} v + \rho \dot{v}, \quad (27)$$

where ρ is the mass density of homogeneous pore fluid and v is the apparent volume fraction of fluid. The fractional rate of fluid volume change $\rho \dot{v}^{-1}$ is related to the pore pressure rate by

$$\dot{\rho} \rho^{-1} = \dot{p}/K_f, \quad (28)$$

where K_f is the bulk modulus of the pore fluid. The rate of change of the apparent fluid volume fraction \dot{v} can be expressed as the sum of an elastic portion \dot{v}^e and an inelastic portion \dot{v}^p . Rice³⁶ has shown that under the same circumstances that $\sigma_{ij} + p\delta_{ij}$ is the appropriate form of the effective stress for inelastic deformation, $\dot{v}^p = D_{kk}^p$. Thus, the expression for the inelastic portion is obtained from Eq. (13) as

$$\dot{v}^p = \beta [\sigma'_{kl} \bar{\sigma}_{kl}/2\bar{\tau} + \mu(\dot{p} + \dot{\sigma}_{kk}/3)]/h, \quad (29)$$

where the form of the effective stress appropriate for inelastic deformation has been substituted. For an isotropic material, the

elastic portion \dot{v}^e must have the form

$$\dot{v}^e = A \dot{\sigma}_{kk} + B \dot{p}, \quad (30)$$

where A and B are constants. These coefficients can be obtained using reciprocal relations derived by Biot³⁷ from the existence of a potential function for elastic deformation. The resulting expression for \dot{v}^e is

$$\dot{v}^e = (1/K - 1/K'_s) (\dot{p} + \dot{\sigma}_{kk}/3) - v\dot{p}/K'_s, \quad (31)$$

where K'_s is another bulk modulus that can, again under modest restrictions, be identified with the bulk modulus of the solid constituents. Substituting Eqs. (28)–(31) into Eq. (27) yields

$$\frac{\dot{m}}{\rho} = \dot{p}v(1/K_f - 1/K'_s) + (\dot{p} + \dot{\sigma}_{kk}/3) (1/K - 1/K'_s) + \beta [\sigma'_{kl} \bar{\sigma}_{kl}/2\bar{\tau} + \mu(\dot{p} + \dot{\sigma}_{kk}/3)]/h. \quad (32)$$

The constitutive formulation is completed by Darcy's law which, in the absence of body forces, takes the form

$$q_i = -\rho \kappa \partial p / \partial x_i, \quad (33)$$

where q_i is the mass flow rate in the x_i direction per unit area and κ is a permeability. The permeability is often expressed as $\kappa = k/\mu$, where μ is the fluid viscosity and k has the dimensions of length squared (usually measured in millidarcies, where 1 md = 10^{-11} cm²) or as $\kappa = \bar{k}/\rho g$, where g is the acceleration due to gravity and \bar{k} has the dimensions of velocity.

Equations (26), (32), and (33) must be combined with field equations expressing equilibrium, compatibility, and fluid mass conservation. The last of these can be written as

$$\dot{m} + \partial q_k / \partial x_k = 0. \quad (34)$$

Substituting Eqs. (32) and (33) into Eq. (34) yields a nonhomogeneous, nonlinear diffusion equation for the pore fluid pressure which is coupled to the constitutive equation (26).

The formulation described here has been used in analysis of the effects of coupling between deformation and pore fluid diffusion on earthquake precursory processes.^{2,38} To illustrate one effect of this coupling, it is convenient to specialize to effectively incompressible solid and fluid constituents, that is, $K_f, K'_s, K'_s \gg K$. Such

a simplification is seldom appropriate for brittle rocks but is often used for soils. In this case Eq. (32) reduces to

$$\dot{m}/\rho = \beta \sigma'_{kl} \bar{\sigma}'_{kl}/(2\bar{\tau}h) + (\dot{p} + \dot{\sigma}_{kk}/3)(\beta\mu/h + 1/K) \quad (35)$$

and the effective stress for both elastic and inelastic deformation is $\sigma'_{ij} + p\delta_{ij}$.

When deformation is too rapid to allow time for fluid mass to diffuse from material elements (but still slow enough to allow for pore fluid pressure equilibration among neighboring fissures) the response is said to be undrained. The rate of change of pore fluid pressure during undrained response is obtained by setting the left-hand side of Eq. (35) equal to zero and solving for \dot{p} . The result is

$$\dot{p} = -\dot{\sigma}_{kk}/3 - \beta K \sigma'_{kl} \bar{\sigma}'_{kl}/2\bar{\tau}(h + \mu\beta K). \quad (36)$$

Substituting into Eq. (26) (with $K/K_s \ll 1$) yields

$$2D_{ij} = \bar{\sigma}'_{ij}/G + \sigma'_{ij}(\sigma'_{kl} \bar{\sigma}'_{kl}/2\bar{\tau}^2)/(h + \mu\beta K). \quad (37)$$

Because the solid and fluid constituents have been assumed to be incompressible, $D_{kk} = 0$. Note that the modulus h governing inelastic shearing has been augmented by the term $\mu\beta K$. Hence, for $\beta > 0$ the coupling between pore fluid diffusion and deformation causes the response for rapid undrained deformation to be stiffer than that for slow drained ($\dot{p} = 0$) deformation and the rock mass is said to be dilatantly hardened (Fig. 6). Rice² has shown, how-

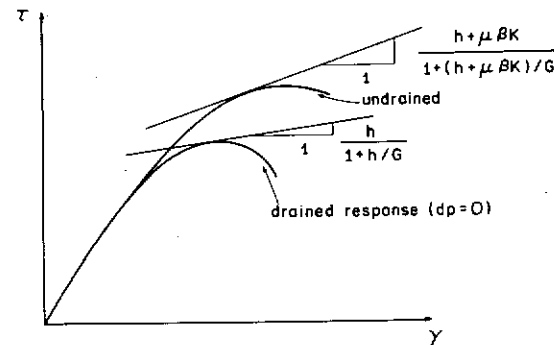


Fig. 6. Dilatantly hardened response to undrained deformation. Solid and fluid constituents are assumed to be separately incompressible.

ever, that the stability of homogeneous dilatantly hardened deformation to the development of localized shearing is governed by the underlying drained behavior. When conditions for the localization of deformation² are satisfied in terms of the underlying drained response, small spatial nonuniformities grow exponentially in time. If $\beta < 0$, as appropriate for material that inelastically compacts under shear, then the undrained short-time response is less stiff than the long-term drained response.

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